Name (IN CAPITALS): **Version #1**

Instructor: Dora The Explorer

Math 10550 Exam 1
Sept. 26, 2024.

- The Honor Code is in effect for this examination. All work is to be your own.
- Please turn off (and Put Away) all cellphones, smartwatches and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)
11.	(•)	(b)	(c)	(d)	(e)
12.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
13.	
14.	
15.	
16.	
Total	

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1.	(a)	(b)	(c)	(d)	(e)
2	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice ________________

13. ________________

14. ________________

15. ________________

16. ________________

Total ________________

Multiple Choice

1.(7pts) Consider the piecewise defined function:

$$p(x) = \begin{cases} x^2 & \text{if } -\infty < x \leq -2 \\ 2x+1 & \text{if } -2 < x < 2 \\ \cos(\pi x) & \text{if } 2 \leq x < \infty \end{cases}$$

Which of the following gives the value of $(p \circ p)(-2) = p(p(-2))$?

Solution: We have, $p(-2) = (-2)^2 = 4$, and $p(4) = \cos(4\pi) = 1$, so

$$(p \circ p)(-2) = p(p(-2)) = p(4) = 1.$$

(a) 1 (b) -1 (c) 0 (d) 9 (e) 4

2.(7pts) An object moving in a straight line has position function $s(t) = t^3 - t^2 - 5$, where time, t, is measured in seconds and and $s(t)$ gives the distance (in feet) of the object from the origin at time t. What is the average velocity of the object over the time interval from $t = 0$ to $t = 2$?

Solution:

$$\text{average velocity} = \frac{s(2) - s(0)}{2 - 0} = \frac{(8 - 4 - 5) - (0 - 0 - 5)}{2} = \frac{-1 - (-5)}{2} = \frac{4}{2} = 2$$

(a) 2 ft/s (b) 4 ft/s (c) -3 ft/s (d) 3 ft/s (e) 8 ft/s

3.(7pts) Evaluate the following limit:

$$\lim_{x \to 2^+} \frac{\sqrt{x^2+5}-3}{x-2}.$$

Solution: We start by simplifying the expression,

$$\frac{\sqrt{x^2+5}-3}{x-2} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \frac{x^2+5-9}{(x-2)(\sqrt{x^2+5}+3)} = \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+5}+3)} = \frac{x+2}{\sqrt{x^2+5}+3}$$

The limit is now,

$$\lim_{x \to 2^+} \frac{\sqrt{x^2+5}-3}{x-2} = \lim_{x \to 2^+} \frac{x+2}{\sqrt{x^2+5}+3} = \frac{4}{\sqrt{9}+3} = \frac{4}{6} = \frac{2}{3}.$$

(a) $\frac{2}{3}$ (b) 1 (c) 4 (d) ∞ (e) $\frac{1}{2}$

4.(7pts) Compute

$$\lim_{x \to 1^-} \frac{x^2+x-2}{x^2-2x+1}.$$

Solution: We start by simplifying the expression,

$$\frac{x^2+x-2}{x^2-2x+1} = \frac{(x+2)(x-1)}{(x-1)^2} = \frac{x+2}{x-1}.$$

For $x < 1$ and $|x-1|$ almost 0, we have that $(x+2) > 0$ and $(x-1) < 0$. Putting this all together, we have that

$$\lim_{x \to 1^-} \frac{x^2+x-2}{x^2-2x+1} = \lim_{x \to 1^-} \frac{x+2}{x-1} = -\infty.$$

(a) $-\infty$ (b) 0 (c) ∞

(d) 1 (e) Does not exist and is not $+\infty$ or $-\infty$

5.(7pts) Let

$$f(x) = \begin{cases} 2m\cos(\pi x) + x + 1 & x \geq 1 \\ x + 5 & x < 1 \end{cases}$$

What value of the constant m will make the function $f(x)$ continuous?

Solution: In order for the function $f(x)$ to be continuous at the point $x = 1$ we must have,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$

Using the definition of f, this equality becomes,

$$6 = \lim_{x\to 1^-} x + 5 = \lim_{x\to 1^+} 2m\cos(\pi x) + x + 1 = -2m + 2.$$

Solving for m we have, $2m = 2 - 6 = -4$, and so $m = -2$.

(a) $m = -2$

(b) $m = 2$

(c) No such value of m exists.

(d) $m = 0$

(e) $m = 1/2$

6.(7pts) Let

$$f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$$

Which of the following gives the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$? **Solution:**

$$f(x) = \sqrt{x} - \frac{4}{\sqrt{x}} = x^{\frac{1}{2}} - 4x^{\frac{-1}{2}},$$

so

$$f'(x) = \frac{1}{2}x^{\frac{-1}{2}} - (4)\left(\frac{-1}{2}\right)x^{\frac{-3}{2}} = \frac{1}{2}x^{\frac{-1}{2}} + 2x^{\frac{-3}{2}}.$$

Thus,

$$f'(1) = \frac{1}{2}(1) + 2(1) = \frac{5}{2}.$$

Further,

$$f(1) = 1 - \frac{4}{1} = -3,$$

so we need to find the equation of a line with slope $m = \frac{5}{2}$ and point $(1, -3)$. We obtain,

$$y - (-3) = \frac{5}{2}(x - 1)$$

$$y = \frac{5}{2}x - \frac{5}{2} + (-3) = \frac{5}{2}x - \frac{11}{2}$$

(a) $y + 3 = \frac{5}{2}(x - 1)$ or $y = \frac{5}{2}x - \frac{11}{2}$

(b) $y + 3 = -\frac{3}{2}(x - 1)$ or $y = -\frac{3}{2}x - \frac{3}{2}$

5. Initials: ____________

(c) $y - 3 = \frac{5}{2}(x - 1)$ or $y = \frac{5}{2}x + \frac{1}{2}$

(d) $y - 3 = -\frac{3}{2}(x - 1)$ or $y = -\frac{3}{2}x + \frac{9}{2}$

(e) $y + 3 = -3(x - 1)$ or $y = -3x$

7.(7pts) If

$$f(x) = \frac{x^2+1}{x^{10}+1},$$

find $f'(1)$.

Solution: To compute $f'(x)$, we may, for example, apply the quotient rule. This yields,

$$f'(x) = \frac{(2x)(x^{10}+1) - (x^2+1)(10x^9)}{(x^{10}+1)^2}$$

Evaluating this at $x = 1$,

$$f'(1) = \frac{2(2) - 2(10)}{2^2} = \frac{-16}{4} = -4.$$

(a) -4 (b) 6 (c) 24 (d) -16 (e) $\frac{1}{5}$

8.(7pts) Compute

$$\lim_{x\to 0} \frac{\sin(2x)}{\cos^3(x)\sin(4x)}.$$

Solution: The key idea is to use the formula

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin(x)}$$

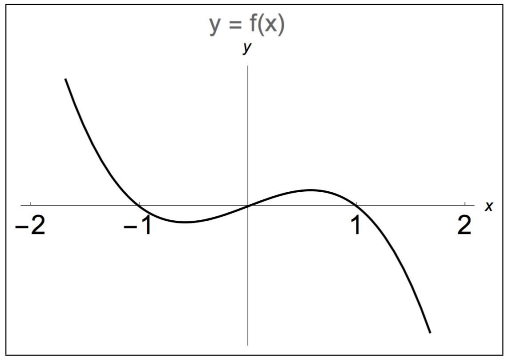
to be able to evaluate the limit as a product of limits.

$$\lim_{x\to 0} \frac{\sin(2x)}{\cos^3(x)\sin(4x)} = \lim_{x\to 0} \frac{1}{\cos^3(x)} \cdot \frac{\sin(2x)}{2x} \cdot \frac{4x}{\sin(4x)} \cdot \frac{2}{4}$$

$$= \frac{1}{2} \cdot (\lim_{x\to 0} \frac{1}{\cos^3(x)}) \cdot (\lim_{x\to 0} \frac{\sin(2x)}{2x}) \cdot (\lim_{x\to 0} \frac{4x}{\sin(4x)}) = \frac{1}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{1}{2}$$

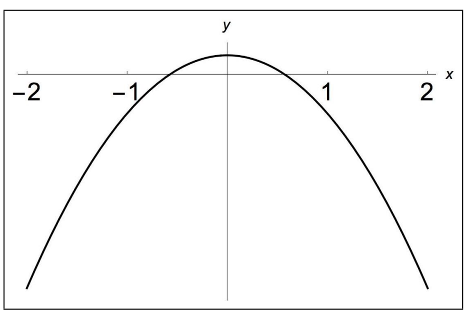
(a) $\frac{1}{2}$ (b) 2 (c) ∞ (d) -2 (e) 0

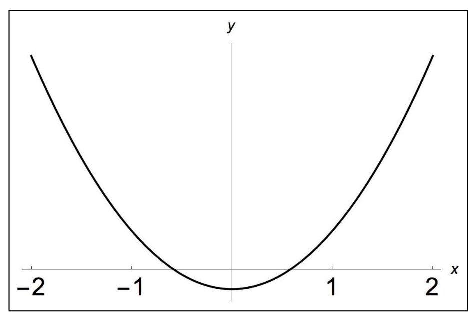
9.(7pts) The graph of $f(x)$ is shown below:

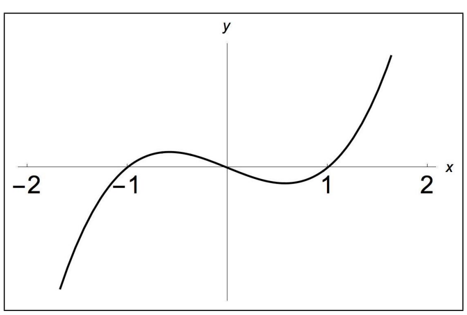


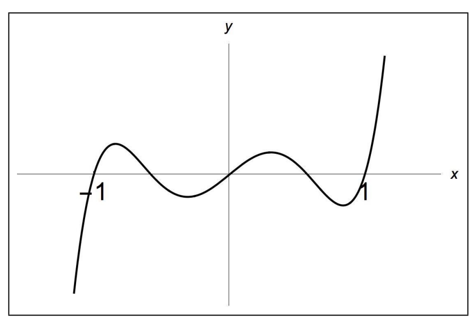
Which of the following is the graph of $f'(x)$?

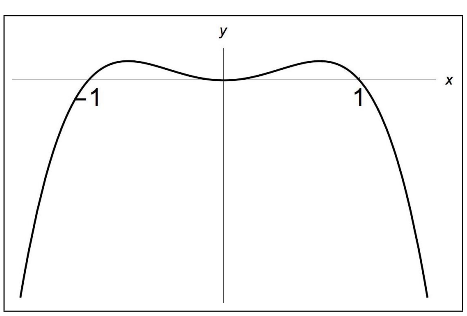
Solution: Given the graph of f, we start by finding values of x where $f'(x) = 0$ - that is, points x where the slope of the tangent line to $f(x)$ is zero. This is *roughly* at $x = -1/2, 1/2$. This rules out options c), d), and e), for, they all cross the x-axis at either the wrong points, or at too many points. Finally, notice that the slope of f is negative for $x < -1$, so b) is ruled out, and we are left with a). Alternatively, note that $f(x)$ is approximately $-x^3$, and so its derivative is approximately $-3x^2$, that is, a downward facing parabola.

(a)


(b)


(c)


(d)


(e)


10.(7pts) A fly is moving on a straight line with position at time t is given by

$$s(t) = t\cos(\pi t),$$

where time, t, is measured in seconds and $s(t)$ gives the distance (in feet) of the fly from the origin at time t. What is the acceleration of the fly when $t = 1$?

Solution: We need to find the second derivative of $s(t)$ and evaluate at $s = 1$. We will need to use product and chain rules to find $s'(t)$ and $s''(t)$.

$$s'(t) = (1)\cos(\pi t) + t\cdot(-\pi\sin(\pi t)) = \cos(\pi t) - \pi t\sin(\pi t)$$

$$s''(t) = -\pi\sin(\pi t) + (-\pi\sin(\pi t) - \pi t\cdot\pi\cos(\pi t)) = -2\pi\sin(\pi t) - \pi^2 t\cos(\pi t)$$

$$s''(1) = 0 - \pi^2(1)(-1) = \pi^2$$

(a) π^2 ft/s^2

(b) $-\pi$ ft/s^2

(c) 1 ft/s^2

(d) -1 ft/s^2

(e) $(-1-\pi)$ ft/s^2

11.(7pts) Compute $f'(x)$ if

$$f(x) = \sqrt{1+\sin^2 x}.$$

Solution:

We use the chain rule, this yields,

$$f'(x) = \frac{1}{2}\left(\sqrt{1+\sin^2 x}\right)^{-1/2}\frac{d}{dx}(1+\sin^2(x)) = \frac{1}{2}\left(\sqrt{1+\sin^2 x}\right)^{-1/2} 2\sin(x)\cos(x)$$

That is,

$$f'(x) = \frac{\sin(x)\cos(x)}{\sqrt{1+\sin^2 x}}$$

(a) $\dfrac{\sin x\cos x}{\sqrt{1+\sin^2 x}}$

(b) $\dfrac{\sin x}{\sqrt{1+\sin^2 x}}$

(c) $\dfrac{\cos^2 x}{2\sqrt{1+\sin^2 x}}$

(d) $\dfrac{1}{2\sqrt{2\sin x\cos x}}$

(e) $\sin x\cos x\sqrt{1+\sin^2 x}$

12.(7pts) Consider the following table of function values:

	$x = 1$	$x = 2$
$f(x)$	-1	5
$f'(x)$	1	3
$g(x)$	2	1
$g'(x)$	1/2	-1

Find $\frac{d}{dx}\left(\frac{1}{f(g(x))}\right)$ when $x = 1$.

Solution: Using chain rule we obtain that

$$\frac{d}{dx}\left(\frac{1}{f(g(x))}\right) = \frac{d}{dx}\left((f(g(x))^{-1}\right) = (-1)f(g(x))^{-2}2 \cdot f'(g(x) \cdot g'(x) = \frac{-f'(g(x)) \cdot g'(x)}{f(g(x))^2}$$

Evaluating at $x = 1$, we obtain

$$\frac{-f'(2) \cdot \frac{1}{2}}{(f(2))^2} = \frac{-(3)(\frac{1}{2})}{5^2} = \frac{-3}{50}$$

(a) $-\frac{3}{50}$ (b) $\frac{2}{3}$ (c) $-\frac{3}{10}$ (d) $\frac{3}{25}$ (e) 75

Partial Credit

For full credit on partial credit problems, make sure you justify your answers.

13.(10pts) Show that there is at least 1 solution to the equation

$$x^3 + 2x - 2 = 0$$

in the interval $0 \leq x \leq 1$. Justify your answer and identify any theorems you use.

Solution: The function $f(x) = x^3 + 2x - 2$ is continuous, being a polynomial in x. On one hand, $f(0) = -2$, on the other hand, $f(1) = 1 + 2 - 2 = 1$, so by the Intermediate Value Theorem there exists some $c \in (0, 1)$ for which $f(c) = 0$, that is, there exists at least 1 solution to the equation in the interval $[0, 1]$.

14.(12pts) (a) Find the derivative of

$$f(x) = \frac{1}{2x-1}$$

using the limit definition of the derivative.

Solution: Recall the limit definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Applying this to $f(x) = \frac{1}{2x-1}$, we obtain that

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{2(x+h)-1} - \frac{1}{2x-1}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{2x-1}{(2(x+h)-1)(2x-1)} - \frac{2(x+h)-1}{(2(x+h)-1)(2x-1)}\right)$$

$$= \lim_{h\to 0} \frac{2x-1-2x-2h+1}{h(2(x+h)-1)(2x-1)} = \lim_{h\to 0} \frac{-2h}{h(2(x+h-1))(2x-1)} = \lim_{h\to 0} \frac{-2}{(2(x+h)-1)(2x-1)}$$

$$= \frac{-2}{(2x-1)^2}$$

(b) Verify your answer to part (a) using the formulas for differentiation.

Solution:

$$f(x) = \frac{1}{2x-1} = (2x-1)^{-1},$$

so by chain rule we obtain that $f'(x) = -(2x-1)^{-2} \cdot 2 = \frac{-2}{(2x-1)^2}$

True-False.

15.(6pts) Please circle "TRUE" if you think the statement is true, and circle "FALSE" if you think the statement is False.

(a)(2 pts.: 1 pt for answer, 1 pt. for justification(no half points).) The slope of the tangent line to the curve $y = \cos x$ at $x = \dfrac{\pi}{3}$ is $\dfrac{\sqrt{3}}{2}$.

Solution: The slope of the tangent line to $y = \cos(x)$ at $x = \pi/3$ is given by $-\sin(\pi/3) = -\sqrt{3}/2$.

TRUE **FALSE**

(b)(1 pt. No Partial credit) If the function $f(x)$ is continuous at $x = a$, then it must be differentiable at $x = a$.

Solution: This is false, the absolute value function is the standard counterexample - it is continuous everywhere, but not differentiable at $x = 0$.

TRUE **FALSE**

(c))(2 pts.: 1 pt for answer, 1 pt. for justification(no half points).)

If $f(x) = 4x^3 + 2x^2 + x + 1$, then $f''(1) = 30$.

Solution:

$$f'(x) = 12x^2 + 4x + 1, \text{ so } f''(x) = 24x + 4$$
$$\text{Thus, } f''(1) = 24 + 4 = 28$$

TRUE **FALSE**

(d))(1 pt. No Partial credit) $\dfrac{\pi}{2}$ is in the domain of the function $f(x) = \tan(x)$.

Solution: It is not in the domain as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\cos(\frac{\pi}{2}) = 0$

TRUE **FALSE**

 Initials: ____________

16.(2pts) You will be awarded these two points if you write your name in CAPITALS on the front page and you mark your answers on the front page with an X through your answer choice like so: ~~(a)~~ (<u>not</u> an O around your answer choice) .

Initials: ____________

ROUGH WORK